

Static and Dynamical Anisotropy Effects in Mixed State of D - Wave Superconductors

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Abstract

We describe effects of anisotropy caused by the crystal lattice in d-wave superconductors with s-wave mixing using the effective free energy approach. Only the d-wave order parameter field d is introduced, while the effect of the s wave mixing as well as other effects breaking the rotational symmetry down to the fourfold symmetry of the crystal are represented by a single four (covariant) derivative term: $\eta d^*(\Pi_y^2 - \Pi_x^2)^2 d$. The single vortex solution in a phenomenologically interesting range of parameters is almost identical to the two order parameters approach. We analytically consider the most general oblique lattice and orientation, but find that only rectangular body centered lattices are realized. A critical value η_c at which a phase transition from the rectangular lattice to the square lattice takes place. The influence on the phase transition line is discussed. The formalism is extended to the time dependent

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anisotropic Ginzburg-Landau equations in order to calculate the effect of the anisotropy on the flux flow. The moving vortex structure is established and the magnetization as function of the current is calculated. Although the linear conductivity tensor is rotationally invariant due to the fourfold symmetry, the nonlinear one shows anisotropy. We calculate dependence of both direct and Hall I-V curves on the angle between the current and the crystal lattice orientation.

I. INTRODUCTION AND SUMMARY

It is widely believed that the superconductivity in layered high T_c cuprates is largely due to the $d_{(x^2-y^2)}$ pairing [1]. The evidence for the d - wave pairing comes from variety of different measurements. A partial list includes the μ SR measurements of the penetration depth [2], quantum phase interference [3], angular resolved photoemission [4], thermal conductivity [5], vortex lattice structure observed using neutron scattering [6] and tunneling spectroscopy [7] and nuclear spin relaxation rate [8]. While most of these experiments directly probe the energy gap and the low lying excitations, the vortex lattice observations are different. They try to relate the structure and interactions of Abrikosov vortices to the nature of the order parameter. This would be extremely difficult if the order parameter would be a pure d wave. The structure of a single vortex and even the vortex lattice on the level of the "macroscopic" Ginzburg - Landau equations would be the same as for that of the usual s wave superconductors.

There are numerous indications, both theoretical [10,9] and experimental [3,8], that even though the major pairing mechanism is the d wave pairing, there is a small admixture of the s wave pairs in the condensate. Ren et al [11] using a phenomenological microscopic model (in the weak-coupling limit) and Soininen et al [12] considering attractive nearest neighbors interaction derived a two field effective Ginzburg-Landau (GL) type theory. The two complex order parameter fields, s and d describe the gap function in corresponding

channels. The most general free energy of this kind reads [13,14]:

$$f = \alpha_s |s|^2 - \alpha_d |d|^2 + \beta_1 |s|^4 + \beta_2 |d|^4 + \beta_3 |s|^2 |d|^2 + \beta_4 (s^{*2} d^2 + d^{*2} s^2) \\ + \gamma_s |\mathbf{\Pi} s|^2 + \gamma_d |\mathbf{\Pi} d|^2 + \gamma_v [s^* (\Pi_y^2 - \Pi_x^2) d + c.c.] \quad (1)$$

where $\mathbf{\Pi} \equiv -i\nabla - e^* \mathbf{A}$ is the covariant derivative and e^* is the charge of the Cooper pair (throughout this paper we use the convention $c = \hbar = 1$). Within a particular microscopic model there might be some relations between these coefficients, but since the ultimate microscopic theory is not known as yet, all of them should be considered as phenomenologically fixed parameters.

Using equations following from this free energy or more fundamental equations (see recent quasiclassical Eilenberger equations treatment in [15]), one obtains a characteristic four-lobe structure with four zeros for the s - wave inside a single vortex [13,16]. Therefore the vortex core loses the full rotational symmetry and only the fourfold D_{4h} symmetry remains. The distribution of the magnetic field was also obtained recently [16,13]. A somewhat different structure, however, was found in another recent numerical work [18] (our analytic calculation confirms that of [16,13] and contradicts that of [18]).

Outside the core the s-wave vanishes, while the d-wave becomes rotationally invariant, indistinguishable from the usual Abrikosov solution. Therefore, to look for differences in the behavior of vortices one would like to be closer to H_{c2} , so that the core will be more important (which is not easy for high T_c superconductors due to their large H_{c2}). However, since the fourfold vortex core structure comes into a conflict with the high symmetry of the triangular lattice, the asymmetry of vortices can distort the usual triangular vortex lattice at already accessible fields much lower than H_{c2} . Another phenomenon in which the vortex core plays a major role is the dissipation in the course of the flux flow. These are the two major phenomena in clean superconductors in which one can look for anisotropy effects (we neglect pinning and other disorder and fluctuations, and concentrate only on YBCO single crystals).

In this paper we study in detail the above two phenomena, vortex and the vortex lattice

structure and the flux flow, using a greatly simplified model: time independent and time dependent one component effective Ginzburg - Landau equations. This formulation, essentially without loss of generality, allows us to avoid numerical methods and to extend and clarify various delicate questions about the single vortex and the vortex lattice structure for which there is some controversy or lack of concrete proof. In particular the location of the phase transition to the square lattice is calculated and depends just on one combination of the parameters. Moreover the quantitative discussion can be extended to study moving flux lattices, which, as is well known [19,20], are much more demanding, as far as the calculational complexity is concerned. Their shape and orientation of moving lattices (or large bundles) with respect to the crystal lattice and electric field (or current) is determined. Then we calculate the current due to the flux flow as function of the external electric field. Since the deviations from the full rotational symmetry in the flux flow can be clearly seen only in the nonlinear regime (the fourfold symmetry forces the full rotational invariance of the Ohmic conductivity tensor) one has to go beyond linear response.. The simplified formulation is indispensable in this case [19,20], but the result turns out to be remarkably simple. In the rest of this section we outline the basic assumptions, methods and results pointing out where in the paper more details can be found.

Within the two field formulation Soininen et al [13] observed that in predominantly d - wave superconductor the s - wave component is generally very small: it is "induced" by variations of the larger d component. Mathematically the dominance of the d - wave follows from the fact that coefficient of the d^*d term, $-\alpha_d = \alpha'(T - T_c)$, is negative, while that of the s^*s , α_s , is positive in the free energy, Eq.(1). Therefore, it is the d field which acquires a nonzero value. Then the rotationally noninvariant derivative term $s^*(\Pi_y^2 - \Pi_x^2)d$ "communicates" the deviations from the condensate value of d to s . Note that this is the only term up to (scaling) dimension three in the free energy which breaks the full rotational symmetry. If its coefficient γ_v is not very large, the s field never becomes comparable to d . Soininen et al [13] observed that even near the core, where d is the smallest, it is nevertheless larger than s by a factor of 20 at least. This in particular means that many small terms

like $|s|^4$ are irrelevant. The field d is the critical field near T^c , while s is not and can be "integrated out" perturbatively. It will be explained in some detail in Section II, that this generates an effective (scaling) dimension five term for d , breaking the rotational symmetry, so that our starting point to study the rotation symmetry breaking effects is an effective free energy

$$f_{eff}[d] = \frac{1}{2m_d} |\Pi d|^2 - \alpha_d |d|^2 + \beta |d|^4 - \eta d^* (\Pi_y^2 - \Pi_x^2) d. \quad (2)$$

Here we have replaced γ_d by a more conventional notation $1/2m_d$ and parameter $\eta \equiv \gamma_v^2/\alpha_s$ quantifies the deviation from the rotational symmetry. Its relation to other parameters in the two field approach is derived in Section II. We calculate all the above mentioned rotational symmetry breaking effects to the first order in η . This formulation follows the general philosophy of effective free energy written in terms of critical fields only. The noncritical fields just renormalize the coefficients. The rotational symmetry breaking term has dimension five, but breaks the symmetry and is therefore a "dangerous irrelevant term" using the terminology of critical phenomena [21]. The contributions to it might come not only from the d - s mixing, which always give positive η , but also from other sources. Even in conventional superconductors such effects exist [22]. In YBCO, twinning might be an important contribution to it. This formulation avoids the problem of the second phase transition at T_s that one encounters assuming $\alpha_s = \alpha'(T_s - T)$ in the two field formalism.

The single vortex solution is obtained in Section III. It is almost identical to the solutions obtained earlier within the framework of the two order parameters theory. One still can define the "effective s-wave field" by $s = -\frac{\gamma_v}{\alpha_s} (\Pi_y^2 - \Pi_x^2) d$ and observe the four lobe structure, see Fig. 2 and 3 for d and s components respectively. Relation to earlier work (discrepancies or common points) are summarized in the Appendix.

The vortex lattice near H_{c2} is studied comprehensively in Section IV. The simplicity of the formulation allows for an analytic study of all the possibilities, not considered before or considered using uncontrollable approximations. The degrees of freedom we include in the analysis contain: (1) an arbitrary rotation angle φ between the crystal lattice and the vortex

lattice (Fig. 4), and (2) all the possible lattice, not only the rectangular ones considered before ([12], [13], [18]). Instead of using the variational method to solve the linearized set of equations, we solve it exactly even in the moving lattice case. This is the first time that the lattice is demonstrated to be rectangular body centered using the most general lattice in the analysis. We tabulate the lattice characteristics for different η in Fig. 7. At certain value of η there is a phase transition from rectangular to a more symmetric square lattice first noticed [13]. The existence of a phase transition becomes obvious in our formulation in which the effective strength of the four fold symmetry is proportional to the magnetic field, characterized by a dimensionless parameter $\eta' \equiv \eta m_d e^* H$. In low fields, the four fold symmetry is subdominant, so the lattice is closer to the triangle lattice. In high fields, the four fold symmetry dominates, so the lattice becomes square. We find that the transition occurs at $\eta'_c = .0235$. The upper critical field line $H_{c2}(T)$ is given in Eq.(27). However at the end of this Subsection IV.C we caution against too direct interpretation of this result by considering other possible effects which are isotropic and also contribute to the curvature.

The moving lattice solutions are derived in Section V from a time dependent generalization of Eq.(2). They are not only needed for the nonlinear conductivity calculation which follows, but are also interesting in their own right, since they are, in principle, observable. In the one field formalism there is only one additional parameter to be added to take the dissipation effects into account: the coefficient of the time derivative term of d . Unlike in the case of the pure s-wave superconductor, the moving (with arbitrary, not infinitesimal, velocity) lattice solution in this case can not be obtained from the static one by a simple Galilean boost [20]. It is a nontrivial problem and we were able to solve it perturbatively in η . Unlike the s - wave moving lattices (which are triangular [20]), orientation is determined by the direction of the crystal lattice as well as by the current direction. The "dynamical phase transition line as function of current and its orientation with respect to the atomic lattice are quantitatively discussed in Subsection V.B and the result is given in Eqs.(62,63,64). Magnetization close to H_{c2} (or the Abrikosov β_A) is a very simple function of the current J and its direction. The result for the Abrikosov β_A is:

$$\beta_A(\mathbf{E}) = \beta_A^0 + c\eta E^2 |\cos 2\Theta|, \quad (3)$$

where Θ is an angle between the electric field \mathbf{E} and an axis of the underlying atomic lattice axis. The number β_A^0 (typically a bit larger than 1) is the usual Abrikosov β parameter for a given lattice defined in Eq.(41) and the constant c is given in Subsection V.C (Eqs.(75,76)).

Corrections to the linear conductivity tensor (which cannot break the rotational symmetry) are briefly discussed and a detailed calculation of the effect of the anisotropy on the nonlinear flux flow are given in Section VI. The result is remarkably simple. In addition to isotropic linear part there is an anisotropic cubic in E term is:

$$\Delta \mathbf{J} = \eta \frac{2m_d \gamma^3 E^3}{\beta_A^0 e^* H^4} (1 + \cos 4\Theta) \quad (4)$$

and the Hall current is

$$\Delta \mathbf{J}_{Hall} = -\eta \frac{2m_d \gamma^3 E^3}{\beta_A^0 e^* H^4} \sin 4\Theta \quad (5)$$

The two nonlinear contributions to currents are simply related. In these expressions γ is the coefficient of the time derivative term in time dependent Ginzburg - Landau equation Eq.(10). Both direct and Hall I-V curves depend on the angle between the current and the crystal lattice orientation via the fourth harmonic only. The result, contains only cubic dependance of the currents on the electric field, higher orders being cancelled.

Finally in Section VII we conclude by briefly discussing possible experiments to observe various above mentioned effects, as well as some generalizations.

II. TIME INDEPENDENT AND TIME DEPENDENT EFFECTIVE GINZBURG - LANDAU EQUATIONS.

On very general grounds superconductivity is sometimes considered as a phenomenon of "spontaneous $U(1)$ gauge symmetry breaking " [23] irrespective of the mechanism of pairing or channels in which it occurs. The $U(1)$ electric charge symmetry should be represented by a single order parameter: the superconducting $U(1)$ phase. Mechanisms of superconductivity, as far as connection to the gap functions appearing in the microscopic description

is concerned, may differ, but this general order parameter representation of the superconducting phase remains the same. While in pure s - wave or d - wave superconductors the $U(1)$ phase is simply identified with the phase of the gap function, in more complicated microscopic theories with few channels opened, the superconducting phase is just the common phase of various gap functions. Therefore, quantities other than the phase which enter various phenomenological Ginzburg-Landau (GL) type equations, although useful, are not directly related to the spontaneous gauge symmetry breaking.

The s-d mixing two component GL free energy Eq.(2) leads to the following set of equations:

$$(\gamma_d \Pi^2 - \alpha_d) d + \gamma_v (\Pi_y^2 - \Pi_x^2) s + 2\beta_2 |d|^2 d + \beta_3 |s|^2 d + 2\beta_4 s^2 d^* = 0 \quad (6)$$

$$(\gamma_s \Pi^2 + \alpha_s) s + \gamma_v (\Pi_y^2 - \Pi_x^2) d + 2\beta_1 |s|^2 s + \beta_3 |d|^2 s + 2\beta_4 d^2 s^* = 0 \quad (7)$$

As we discussed in the Introduction, the noncritical s component is induced by the $d - s$ mixing gradient term. Therefore the length scale of the variations of the field s is the d = wave's coherence length $\xi_d = \sqrt{\frac{\gamma_d}{\alpha_d}}$. Consequently the derivative term in Eq.(7) $\gamma_s \Pi^2 s \sim (\gamma_s / \xi_d^2) s$ is small compared to $\alpha_s s$. This requirement $\frac{\gamma_s}{\gamma_d} \frac{\alpha_d}{\alpha_s} \ll 1$ (typically $\frac{\gamma_s}{\gamma_d} \sim 1$) holds for the vortex solution of [13,17,16] and is, in fact, an excellent approximation in both near and far from the core regions. Then, according to Eq.(7), the field s is, to the first order in $1/\alpha_s$:

$$s \approx -\frac{\gamma_v}{\alpha_s} (\Pi_y^2 - \Pi_x^2) d \quad (8)$$

Substituting this equation back to Eq.(6), we obtain, to first order in $1/\alpha_s$,

$$\left(\frac{1}{2m_d} \Pi^2 - \alpha_d \right) d - \eta (\Pi_y^2 - \Pi_x^2)^2 d + 2\beta |d|^2 d = 0 \quad (9)$$

where γ_d was replaced by a more standard notation $\frac{1}{2m_d}$ and $\eta \equiv \frac{\gamma_v^2}{\alpha_s}$. The second term should be treated as a perturbation. In principle there are terms of higher order in $1/\alpha_s$, but one cannot take them consistently into account without simultaneously including additional terms in the original two field GL equations. From Eq.(9) the effective free energy Eq.(2)

follows. Of course the effective free energy f_{eff} Eq.(9) is valid only if possible higher orders of the field d and derivatives are neglected.

Note that even the linearized set of Eq.(6,7) is highly nontrivial. Authors of Ref. [13] resort to the variational estimate to find a solution. On the other hand, the linearized Eq.(9) can be easily solved perturbatively in $1/\alpha_s$. Another advantage of this equation, especially as far as relation to experiments is concerned, is that the number of coefficient is much smaller. Instead of 6 additional parameters in the two field free energy, there is just one additional adjustable parameter compared to the usual s - wave GL equations.

The effective free energy approach can be also motivated by considering the fluctuations of the two fields theory. Assuming that only the d field is critical, one can integrate out the s field fluctuations perturbatively. The lower order terms coming out from this analysis are precisely the same as f_{eff} as expected. In principle, the coefficients of the effective one field free energy should be fixed by a microscopic theory in the same spirit as the way that the coefficients of the two component equations should be fixed. The general form of the effective free energy can be obtained just by the dimensional analysis and symmetry. Generally, we should allow all the terms invariant under the group D_{4h} with dimensions five or less. It is a well known property of the D_{4h} symmetry that at the level of the dimension three (relevant) terms, full rotational symmetry is restored.

Therefore, one has to consider "irrelevant" (scaling) dimension five gradient terms in order to break the rotational symmetry down to D_{4h} . *A priori* there are five such terms: $d^*\Pi_x^4d$, $d^*\Pi_y^4d$, $d^*\Pi_x^2\Pi_y^2d$, $d^*\Pi_x\Pi_y^3d$, and $d^*\Pi_x^3\Pi_yd$. The last two break the reflection symmetry, while out of the remaining three, one can only construct two linear combinations invariant under the rotation of $\pi/2$. They are $d^*(\Pi_y^2 - \Pi_x^2)^2d$ and $d^*(\Pi^2)^2d$. The second term is fully rotational invariant, and therefore is not important for studying anisotropy effects. It, however, contributes to effects not directly related to anisotropy such as the shape of the phase transition line. We will come back to this point in Section IV. Similarly, all the other dimension five terms that one should in principle include are rotationally invariant. They just add a small contribution to the rotationally invariant parts of physical quantities and

need not be included when studying the rotational symmetry breaking effects.

The time dependent GL equation in the one field formulation is also quite simple:

$$\gamma \left(\frac{\partial}{\partial t} + ie^* \Phi \right) d = - \left(\frac{1}{2m_d} \Pi^2 - \alpha_d \right) d + \eta (\Pi_y^2 - \Pi_x^2)^2 d - 2\beta |d|^2 d, \quad (10)$$

where Φ is the electric potential. The above equation describes the time evolution of the order parameter and will be used to describe moving vortex systems. It involves only one additional parameter γ compared to the 2×2 matrix for the two field formalism. Although, in principle, this parameter describing various dissipation effects can have a complex part [19], we will consider only real values.

III. THE SINGLE VORTEX SOLUTION

In this section we find an isolated vortex solution of the one component equation Eq.(9) near H_{c1} . The opposite case of magnetic field close to H_{c2} will be considered in next section. We measure the field in units of the vacuum expectation value $\Psi_0 = \sqrt{\alpha_d/2\beta_2}$, and length in units of the coherence length $\xi_d = 1/\sqrt{2m_d\alpha_d}$. In strongly type II materials (when the Ginzburg-Landau parameter κ is very large), as is the case in high T_c superconductors, we can safely ignore the magnetic field and the dimensionless GL equation becomes:

$$(-\nabla^2 - 1)d - \lambda(\nabla_y^2 - \nabla_x^2)^2 d + |d|^2 d = 0$$

where $\lambda \equiv 4\eta m_d^2 \alpha_d$ is our dimensionless small perturbative parameter.. We solve it perturbatively in λ as follows. Let $d = d_0 + \lambda d_1$, where $d_0 = f_0(r)e^{i\phi}$ is the solution of the standard unperturbed GL equation. Then the first order equation in λ is

$$(-\nabla^2 - 1)d_1 + (2|d_0|^2 d_1 + d_0^2 d_1^*) = (\nabla_y^2 - \nabla_x^2)^2 d_0 \quad (11)$$

The angular dependence of d_1 is easily observed to contain only three harmonics: $e^{-3i\phi}$, $e^{+i\phi}$ and $e^{5i\phi}$. This is consistent with the fourfold symmetry which is built into the theory. We therefore decompose d_1 into combination of these three harmonics:

$$d_1(r, \phi) = f_{-3}(r)e^{-3i\phi} + f_1(r)e^{i\phi} + f_5(r)e^{5i\phi} \quad (12)$$

The equation becomes

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{9}{r^2} + 1 \right) f_{-3} - f_0^2(2f_{-3} + f_5) = -J_{-3}(r) \quad (13)$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{25}{r^2} + 1 \right) f_5 - f_0^2(2f_5 + f_{-3}) = -J_5(r) \quad (14)$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} + 1 \right) f_1 - 3f_0^2 f_1 = -J_1(r) \quad (15)$$

with J_i defined by:

$$(\nabla_y^2 - \nabla_x^2)^2 [f_0(r)e^{i\phi}] = e^{i\phi} J_1(r) + e^{-3i\phi} J_{-3}(r) + e^{5i\phi} J_5(r)$$

As is well known, the analytic expression for f_0 does not exist, however, there are a number of known good approximations. Using one of them [28], $f_0(r) = \frac{r}{\sqrt{r^2 + \xi_v^2}}$, the set of linear equations are then solved numerically (the third equation decouples from the first two). The results are shown in Fig.1. The d - wave configuration is basically indistinguishable from that of the two fields formalism for $\lambda = 0.15$, see Fig. 2.

Note also that within the same approximation and normalization, the s component is:

$$s = \lambda'(\nabla_y^2 - \nabla_x^2)d_0 \quad (16)$$

where $\lambda' = 2\gamma_v m_d(\alpha_d/\alpha_s) = \frac{\lambda}{2\gamma_v m_d}$ is another dimensionless small parameter. It's easy to see that s has the asymptotic behavior

$$\begin{aligned} s &\sim r e^{-i\phi} \quad r \rightarrow 0 \\ s &\sim \frac{1}{r^2} e^{+3i\phi} \quad r \rightarrow \infty \end{aligned} \quad (17)$$

The s field is plotted in Fig.3. The different winding numbers in near and far asymptotic regions give rise to four additional poles in the s component in the intermediate region. This confirms calculations of [13] even though some asymptotic analytic expression used there to obtain the numerical results disagree with ours. The comparison with [13] and [17] is presented in Appendix.

IV. THE VORTEX LATTICE NEAR H_{C2}

In this Section we follow a generalization of the Abrikosov's procedure [24,25] to investigate the structure of the vortex lattice near H_{C2} . One first ignores the non-linear terms in the GL equation and finds a set of the lowest energy solutions $\Psi_{k_n}(x, y)$ of the linearized equation. The lattice solution is constructed as a linear superposition

$$d(x, y) = \sum_n C_n \Psi_{k_n}(x, y) \quad (18)$$

in such a way that it is invariant under the corresponding symmetry group of the lattice. It is well known that the free energy near H_{C2} is monotonic in the Abrikosov's parameter β_A , which is defined by $\beta_A = \langle |d|^4 \rangle / \langle |d|^2 \rangle^2$. In the last step needed for some applications, the overall normalization of the order parameter is variationally fixed by minimizing the free energy including non-linear terms.

A general lattice in 2D can be specified by three parameters a, b and α , where a and b are the two lattice constants, while α is the angle between the two primitive lattice vectors (Fig 4). Flux quantization constrains them, so that $Hab \sin \alpha = \Phi_0$. In the d-wave superconductors the rotational symmetry is broken, therefore the relative orientation of the vortex lattice to the underlying lattice becomes important. Later we will denote φ to be the angle between \vec{a} and one of the axes of the underlying lattice. In Abrikosov's original paper [24] he assumed $C_n = C_{n+1}$ and obtained the square lattice, later Kleiner *et al* generalized the procedure to the case where $C_n = C_{n+2}$. In this way all the rectangular body centered lattices can be included in the analysis. In previous works on d-wave superconductivity ref. [13,17], the same formalism was used, however, this does not include the most general lattice. In this section we follow a more generalized formulation of Ref. [25] which covers all possible lattice types.

In our case we first solve the linearized GL equation perturbatively in the anisotropy parameter η . And then we obtain an analytic expression of β_A as a function of a, b (or α) and φ . Finally, we minimize the free energy analytically with respect to φ and numerically with respect to a, b (or α) to find the lattice structure.

A. The perturbative solution of the linearized GL equations

We start from the effective linearized GL equation Eq.(9)

$$\frac{1}{2m_d}\Pi^2 d - \eta(\Pi_y^2 - \Pi_x^2)^2 d = \alpha_d d, \quad (19)$$

where for later convenience we have moved $\alpha_d d$ to the right hand side. It is important to note that in Eq.(19) we have assumed that the coordinate system and the underlying microscopic lattice coincide. Later it will be convenient to orient the coordinate system (x, y) with the Abrikosov vortex lattice rather the atomic crystal. In general, if the crystal is rotated by an angle φ counterclockwise with respect to the coordinate system, Eq.(19) becomes

$$\frac{1}{2m_d}\Pi^2 d - \eta \left[\cos 2\varphi (\Pi_x^2 - \Pi_y^2) + \sin 2\varphi (\Pi_x \Pi_y + \Pi_y \Pi_x) \right]^2 d = \alpha_d d. \quad (20)$$

It is convenient to introduce dimensionless creation and annihilation operators defined by

$$\begin{aligned} \hat{a} &= \frac{i\Pi_+}{\sqrt{2}} l_H, \\ \hat{a}^\dagger &= \frac{-i\Pi_-}{\sqrt{2}} l_H, \end{aligned}$$

where $\Pi_\pm \equiv \Pi_x \pm i\Pi_y$ and the scaling parameter $l_H = 1/\sqrt{e^* H}$ is the magnetic length. In terms of \hat{a} and \hat{a}^\dagger , Eq.(19) becomes

$$\left[\hat{a}^\dagger \hat{a} + \frac{1}{2} - \eta' (e^{+2i\varphi} \hat{a}^{\dagger 2} + e^{-2i\varphi} \hat{a}^2)^2 \right] d(x, y) = \frac{H^0}{2H} d(x, y). \quad (21)$$

Here dimensionless parameter η' is given by $\eta' = \eta m_d e^* H$. For later convenience, we have defined an unperturbed (conventional) upper critical fields $H^0 \equiv \Phi_0/(2\pi\xi_d^2) = 2m_d\alpha_d/e^*$.

It is convenient to choose the Landau gauge $\mathbf{A} = Hx\hat{y}$, since in this gauge the momentum operator \hat{p}_y commutes with \hat{a} . Therefore we can choose $d(x, y)$ to be an eigenvector of \hat{p}_y with eigenvalue k : $d(x, y) = \exp(iky)\psi_k(x)$. The operators \hat{a} and \hat{a}^\dagger in this gauge become

$$\hat{a} = \frac{1}{\sqrt{2}} \left(\frac{d}{d\tilde{x}} + \tilde{x} \right), \quad (22)$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}} \left(-\frac{d}{d\tilde{x}} + \tilde{x} \right), \quad (23)$$

where $\tilde{x} \equiv (x - x_0)/l_H$ is dimensionless with $x_0 \equiv kl_H^2$. Standard perturbation theory for the Schrödinger type equations gives for the lowest eigenvalue:

$$\frac{H^0}{2H} = \frac{1}{2} - 2\eta' + O(\eta^2), \quad (24)$$

This determines the upper critical field. Note that the relative angle φ does not affect H in the lowest order. We will come back to examine this result more closely later. The corresponding eigenfunction $\psi(x)$ is:

$$\psi(\tilde{x}) \equiv \psi_0 + \eta'\psi_1 = |0\rangle + \eta'\frac{e^{4i\varphi}}{4}\sqrt{4!}|4\rangle + O(\eta^2) \quad (25)$$

$$= \left(\frac{1}{\pi l_H^2}\right)^{1/4} \left[1 + \eta'\frac{e^{+4i\varphi}}{16}H_4(\tilde{x})\right] \exp\left(-\frac{\tilde{x}^2}{2}\right). \quad (26)$$

where $H_4(x)$ is the forth Hermit polynomial.

B. The slope of the upper critical field in d-wave superconductors

From Eq.(24) we obtain that the upper critical field H satisfies

$$H^0 = H - 4\eta m_d e^* H^2.$$

Solving it perturbatively, we get

$$H = H^0 + 4\eta m_d e^* (H^0)^2.$$

Reinstating the temperature dependence of coefficients of the GL equation, in the simplest case $\alpha(T) = \alpha'(T_c - T)$, we finally get

$$H(T) = \frac{2m_d\alpha'}{e^*} \left[(T_c - T) + 8\eta m_d^2\alpha' (T_c - T)^2\right]. \quad (27)$$

We observe that around T_c for a positive η the $H(T)$ curve bends upwards, in agreement with the two field results [14,13]. This effect has been reported in some experiments. However one should be cautioned against taking this too seriously. First, as we will discuss in some detail later, the coefficient η should not necessarily be positive in all the samples. For

example, twinning is expected to give a negative contribution to it. For negative η correction to the curvature changes sign. Second, although in this study we concentrate on the effects of the anisotropy, which is represented by the (scaling) dimension five four derivative term $d^*(\Pi_y^2 - \Pi_x^2)^2 d$, as we discussed in Section II, there are other rotationally invariant terms of the same dimensionality. The second dimension five four derivative term, $\tau d^*(\Pi^2)^2 d$, gives contribution similar to Eq.(27). One does a calculation similar to that we performed for $d^*(\Pi_y^2 - \Pi_x^2)^2 d$ and obtains:

$$H_{c2}(T) = \frac{2m_d\alpha'}{e^*} \left[(T_c - T) + 8m_d^2\alpha'(\eta - \tau) (T_c - T)^2 \right].$$

This means that for positive τ the sign of the correction might be changed. Unlike η which can be fixed by rotation symmetry breaking effects experiments, this rotationally invariant correction is more difficult to estimate phenomenologically.

C. The Abrikosov parameter calculation

Now we proceed to calculate the Abrikosov's $\beta_A \equiv \langle |d|^4 \rangle / \langle |d|^2 \rangle^2$. Here the angular bracket is defined as the average over the 2D volume, i.e., $\langle f \rangle \equiv \frac{1}{V} \int d^2\mathbf{r} f(\mathbf{r})$, and V is the total volume of the system. If the function $f(\mathbf{r})$ is periodic, it is sufficient to calculate the average over one unit cell.

In the gauge we have chosen in the previous section, a generic solution of the linearized equation takes the form $\Psi_k(x, y) = \exp(iky) \psi(x - kl_H^2)$. Periodicity in y -direction (our lattice vector \mathbf{a} by assumption is aligned with this axis, see Fig.4) allows the following linear combinations:

$$d(x, y) = \sum_{n=-\infty}^{n=\infty} C_n \Psi_k(x, y) = \sum_{n=-\infty}^{n=\infty} C_n \exp\left(i\frac{2\pi n}{a}y\right) \psi\left(x - n\frac{2\pi l_H^2}{a}\right), \quad (28)$$

If the second lattice constant is b and it makes an angle α relative to the y axis, the periodicity in \hat{b} direction requires that $d(x - b\sin\alpha, y + b\cos\alpha) = d(x, y)$ (up to a phase). One can achieve it by setting $b\sin\alpha = p\Delta x = p(2\pi l_H^2/a)$ and $C_{n+p} = C_n \exp(i2\pi nb\cos\alpha/a)$, where p

is an integer. For simple Bravais lattices, there is only one vortex in each unit cell. Therefore, we can take $p = 1$. The area of the unit cell is $ab \sin \alpha = \Phi_0/H = 2\pi l_H^2$.

As a result, all C_n can be fixed up to an overall constant, to be fixed later:

$$C_n = \exp \left[2\pi i \frac{b}{a} \cos \alpha \frac{n(n-1)}{2} \right] \quad (29)$$

and the wave function becomes:

$$d(x, y) = \sum_{n=-\infty}^{n=\infty} \exp \left[2\pi i \frac{b}{a} \cos \alpha \frac{n(n-1)}{2} \right] \exp \left[i \frac{2\pi n}{a} y \right] \psi(x - nb \sin \alpha), \quad (30)$$

It is convenient to use new rectilinear coordinates whose axes coincide with the vortex lattice directions (Fig.4). We shall denote them as X and Y . Their relations to the old $x-y$ coordinates are $y = Y + X \cos \alpha$ and $x = -X \sin \alpha$.

The average of $|d|^2$ is then found by integrating $|d|^2$ over $0 < X < b$ and $0 < Y < a$. The integration over Y enforces a delta function and simplifies the double summation to

$$\langle |d|^2 \rangle = \frac{1}{ab \sin \alpha} \sum_{n=-\infty}^{\infty} a \sin \alpha \int_0^b dX |\psi[(-X - nb) \sin \alpha]|^2. \quad (31)$$

The summation over n converts the integration domain into $(-\infty, \infty)$. We thus obtain

$$\langle |d|^2 \rangle = \frac{|C_0|^2}{b \sin \alpha} \int_{-\infty}^{\infty} dx |\psi(x)|^2. \quad (32)$$

A similar manipulation on $\langle |d|^4 \rangle$ leads to

$$\begin{aligned} \langle |d|^4 \rangle &= \frac{1}{ab \sin \alpha} \sum_{n,m,n',m'} a \sin \alpha \delta_{m+m', n+n'} \exp \left[2\pi i \frac{b}{a} \cos \alpha \frac{-m^2 - m'^2 + n^2 + n'^2}{2} \right] \\ &\quad \times \int_0^b dX \psi^*[(-X - mb) \sin \alpha] \psi^*[(-X - m'b) \sin \alpha] \psi[(-X - nb) \sin \alpha] \psi[(-X - n'b) \sin \alpha] \\ &= \frac{1}{b \sin \alpha} \sum_{n,m,n',m'} \delta_{m+m', n+n'} \exp \left[2\pi i \frac{b}{a} \cos \alpha \frac{-m^2 - m'^2 + n^2 + n'^2}{2} \right] \\ &\quad \times \int_{-b \sin \alpha}^0 dx \psi^*(x - mb \sin \alpha) \psi^*(x - m'b \sin \alpha) \psi(x - nb \sin \alpha) \psi(x - n'b \sin \alpha). \end{aligned} \quad (33)$$

Eqs.(32) and (33) are general expressions for $\langle |d|^2 \rangle$ and $\langle |d|^4 \rangle$. We now specialize them to our perturbed d field solution Eq.(25). It is easy to see that the correction to $\langle |d|^2 \rangle$ starts from η'^2 . We found that

$$\langle |d|^2 \rangle = \frac{1}{b \sin \alpha} \left(1 + \frac{3}{2} \eta'^2 \right). \quad (34)$$

The first order term vanishes, because according to Eq.(32), it is proportional to the inner product of ψ_0 and ψ_1 . Since we shall be interested only in $O(\eta')$ corrections, we will drop this second order term.

The calculation of $\langle |d|^4 \rangle$ is more involved even in zeroth order [25]. Because of the presence of the Kronecker delta, there are only three independent summations in Eq.(33).

We choose the summation variables to be

$$\begin{aligned} Z &= n + n' = m + m' \\ N &= n - n' \\ M &= m - m' \end{aligned} \quad (35)$$

Note that the new discrete variables Z, M and N are not completely independent since they have to be either all even or all odd simultaneously. The summation in Eq.(33) then becomes

$$\sum_{m, m', n, n'} \delta_{m+m', n+n'} = \sum_{\text{even } Z} \sum_{\text{even } M} \sum_{\text{even } N} + \sum_{\text{odd } Z} \sum_{\text{odd } M} \sum_{\text{odd } N}. \quad (36)$$

To zeroth order in η , the integrand in Eq.(33), after appropriate rearrangement, has a simple Gaussian form

$$\exp \left[-\frac{2 \sin^2 \alpha}{l_H^2} \left(X + \frac{Z}{2} b \right)^2 \right] \exp \left\{ -\frac{b^2 \sin^2 \alpha}{l_H^2} \left[\left(\frac{M}{2} \right)^2 + \left(\frac{N}{2} \right)^2 \right] \right\}. \quad (37)$$

As before, the summation over Z in Eq.(36) extends the range of the integral over X to $(-\infty, \infty)$, so that the gaussian integral becomes a common factor

$$\int_{-\infty}^{\infty} dX \exp \left(-\frac{2 \sin^2 \alpha}{l_H^2} X^2 \right) = \sqrt{\frac{\pi}{2 \sin \alpha}} \frac{l_H}{\sin \alpha}. \quad (38)$$

Pulling out this factor, we obtain

$$\begin{aligned} \langle |d|^4 \rangle_0 &= \sqrt{\frac{\pi}{2}} \frac{l_H}{b \sin \alpha} \left\{ \sum_{\text{even } M, N} \exp \left\{ 2\pi i \frac{b}{a} \cos \alpha \left[-\left(\frac{M}{2} \right)^2 + \left(\frac{N}{2} \right)^2 \right] - \frac{b^2 \sin^2 \alpha}{l_H^2} \left[\left(\frac{M}{2} \right)^2 + \left(\frac{N}{2} \right)^2 \right] \right\} \right. \\ &\quad \left. + \sum_{\text{odd } M, N} \exp \left\{ 2\pi i \frac{b}{a} \cos \alpha \left[-\left(\frac{M}{2} \right)^2 + \left(\frac{N}{2} \right)^2 \right] - \frac{b^2 \sin^2 \alpha}{l_H^2} \left[\left(\frac{M}{2} \right)^2 + \left(\frac{N}{2} \right)^2 \right] \right\} \right\} \end{aligned} \quad (39)$$

It is convenient to introduce the complex variable [25]

$$\zeta \equiv \frac{b}{a} \exp(i\alpha) \equiv \rho + i\sigma. \quad (40)$$

The coefficients of M^2 and N^2 in Eq.(39) then become $-2\pi i\zeta^*/4$ and $2\pi i\zeta/4$. Finally, we obtain

$$\beta_A^0 = \sqrt{\sigma} \left\{ \sum_{n=-\infty}^{\infty} \left| \exp(2\pi i\zeta n^2) \right|^2 + \left| \sum_{n=-\infty}^{\infty} \exp \left[2\pi i\zeta \left(n + \frac{1}{2} \right)^2 \right] \right|^2 \right\}. \quad (41)$$

The above calculation can be straightforwardly extended to include the perturbation of η . The relevant integral now becomes

$$\int_{-b \sin \alpha}^0 dx \psi_1(x - nb \sin \alpha) \psi_0(x - m \sin \alpha) \psi_0(x - n' \sin \alpha) \psi_0(x - m' \sin \alpha),$$

where ψ_1 is now given by Eq.(25). The correction of β_A in the first order of η' , after some tedious algebra, is:

$$\begin{aligned} \beta_A^1 = \frac{\eta'}{4} \sqrt{\sigma} \operatorname{Re} \left\{ \exp(4i\varphi) \left[\sum_{n'} \exp(-2\pi i\zeta^* n'^2) \right] \left[\sum_n \exp(2\pi i\zeta n^2) (64\pi^2 \sigma^2 n^4 - 48\pi \sigma n^2 + 3) \right] \right. \\ \left. + \left(n \rightarrow n + \frac{1}{2}, n' \rightarrow n' + \frac{1}{2} \right) \right\} \end{aligned}$$

D. Minimization of the free energy and optimal vortex lattice structure

Having calculated the Abrikosov parameter β_A , one finds the vortex structure by minimizing it with respect to φ , ρ , and σ . The minimization with respect to the angle φ between the vortex lattice and the crystal axes is easily done analytically. The general form of β_A is

$$\beta_A(\varphi, \rho, \sigma) = \beta_A^0(\rho, \sigma) + \eta \left[e^{4i\varphi} \delta(\rho, \sigma) + e^{-4i\varphi} \delta^*(\rho, \sigma) \right]. \quad (42)$$

Obviously the minimum of β_A is achieved when

$$\varphi = -\arg[\delta(\rho, \sigma)]/4 \pm \pi/4. \quad (43)$$

The minimum of β_A is

$$\beta_A^{\min}(\rho, \sigma) = \beta_A^0(\rho, \sigma) - |\eta\delta(\rho, \sigma)|. \quad (44)$$

The further minimization of $\beta_A^{\min}(\rho, \sigma)$ is done numerically. In Fig. 5, we show a plot of $\beta_A^{\min}(\rho, \sigma)$ for $\eta' = 0.0193$. Due to the fact that the same vortex lattice might be represented by several sets of (ρ, σ) , it is enough to consider the region $0 < \rho < 1/2$, see discussion in [25]. For every η' , we observed that two degenerate minima appear. One is at $\rho = 1/2, \sigma = 0.663$, and is clearly an rectangular body centered lattice with $\alpha = 53^\circ$. The corresponding φ is zero. Therefore the vortex lattice coincides with the crystal axes. The same result was claimed in [13]. The other minimum is at $\rho = 0.275$ and $\sigma = 0.961$ and $\alpha = 74^\circ$, but with φ equal to 37° . This corresponds to the previous lattice rotated by $\pi/2$. To conclude, we observed rectangular body centered lattices only. The lowest energy state is doubly degenerate. It is interesting to note that the fourfold symmetry of the underlying crystal is not completely broken spontaneously by the static vortex lattice - rotations of π and reflections are symmetries of both rectangular lattices. The η' dependence of α and β_A^{\min} are plotted in Fig. 6 and Fig. 7. We observed that there is a phase transition occurred at $\eta' = 0.0235$ where the lattice goes continuously from rectangular to square.

Note that the calculations described in this section, unlike those for the single vortex, are valid for arbitrary (not only large κ) type II superconductor. Therefore the results can be applied to non high T_c materials as well. Using standard methods, one can take into account variations of the magnetic field. We do not repeat here the standard relations between the free energy and the Abrikosov parameter β_A [25]. One also can calculate corrections to the magnetization curves in a standard way using the β_A calculated here.

Despite the fact that general oblique lattices were considered for the first time for d - wave, our numerical analysis shows that they have higher energy than the rectangular body centered ones. Intuitively in the symmetric case this is understandable because the rectangular lattices are more symmetric. Although for the s - wave superconductors this fact has been established [31], for rotationally non-symmetric superconductors this "argument" is not invalid. We are not aware of any mathematical investigations of this question. Moreover,

when the vortex lattice starts moving, the rotational symmetry is further explicitly broken. As we will see in next section, the general oblique lattices nevertheless are not formed.

V. THE MOVING LATTICE SOLUTION OF THE TIME DEPENDENT GINZBURG-LANDAU EQUATION

In this chapter we generalize the above procedure to find the structure for a moving vortex lattice near H_{c2} (the upper critical field itself being a function of the current, as will be discussed in Subsection V.B). One can consider the motion as caused by electric field \mathbf{E} . The vortex lattice velocity is perpendicular to both electric and magnetic field (assumed not to be tilted for simplicity and taken to be in the $+z$ direction): $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$. For a general direction of the electric field the fourfold symmetry of the system is completely (explicitly) broken. Only for several special directions, along the crystal axes $[1,0,0]$, $[0,1,0]$ or along the diagonal lines $[1,1,0]$ or $[1,\bar{1},0]$ the crystal symmetry is not broken completely, only reduced. Even for the simple s - wave time dependent GL equations (TDGL) the problem of finding the moving lattice solution is nontrivial. However there exists the "Galilean boost" trick [20] to solve the linearized (and sometimes even a nonlinear problem for linear response [19]) problem. As we will see shortly, for the d - wave equations, even the linearized equation does not seem to possess a boosted static solution.

Technically the steps follow those of the static case. First we find a complete set of solutions of the linearized equations using perturbation theory in η . Then we impose the periodicity conditions to construct the vortex lattice wave functions. It is more convenient to perform the first step in the gauge aligned in the direction of the electric field, while for the second step it is preferable to use a gauge aligned in the direction of the vortex lattice. We will combine the two steps using gauge transformation. After the wave function is found, it is straightforward to apply the procedure described in the previous section to minimize the Abrikosov's β_A .

A. The perturbative solution of the linearized TDGL equations

To simplify the presentation, we first assume that the direction of the electric field is special: along the crystalline x (or $[1,0,0]$) direction. In this case the vortices are moving in the negative y direction of the coordinate system. We will return to the general case afterwards. Now we will construct the perturbative solution to the linearized TDGL Eq.(10) which we now write in the following "diffusion equation" form:

$$\gamma \left(\frac{\partial}{\partial t} + ie^* \Phi \right) d = - \left(\frac{1}{2m_d} \Pi^2 - \alpha_d \right) d + \eta \left(\Pi_y^2 - \Pi_x^2 \right)^2 d \quad (45)$$

The vector potential we adopt here is the same as that in section IV, while the electric potential can be chosen to be time and y independent:

$$\Phi = -vHx. \quad (46)$$

In this gauge, the variables t and y trivially separate from x ,

$$d(x, y, t) = \exp(iky) \exp(-\omega t/\gamma) \psi(x),$$

where ω can have an imaginary part: $\omega = \omega_R + i\omega_I$. The equation then reduces to a one dimensional "Schroedinger type" one (note there is an anti - Hermitian dissipation term) :

$$\left\{ \frac{1}{2m_d} \left[\hat{p}_x^2 + \left(k - \frac{x}{l_H^2} \right)^2 \right] - i\gamma v e^* Hx - \alpha_d - \eta \left(\Pi_y^2 - \Pi_x^2 \right)^2 \right\} \psi(x) = \omega \psi(x)$$

Completing the square, rearranging the equation and noting that $\omega_I = -ik\gamma v$ one obtains:

$$\left\{ \frac{1}{2m_d} \left[-\frac{d^2}{dx^2} + \frac{1}{l_H^4} (x - x_0 - igl_H)^2 \right] - \eta \left(\Pi_y^2 - \Pi_x^2 \right)^2 - \alpha_d + \frac{1}{2} \gamma^2 m_d v^2 \right\} \psi(x) \quad (47)$$

$$\equiv \left[\left(\hat{K} - \eta \hat{V} \right) - \alpha_d + \frac{1}{2} \gamma^2 m_d v^2 \right] \psi(x) = \omega_R \psi(x). \quad (48)$$

where a dimensionless quantity g is defined by $g \equiv \gamma m_d v l_H$ and $x_0 = kl_H^2$. The parameter α_d should be adjusted (i.e., changing the temperature) such that the lowest eigenvalue ω_R becomes zero, otherwise one gets runaway solutions. This is nothing but the H_{c2} condition generalized to include arbitrary electric field. The operator \hat{K} defined in Eq.48) is simply

$\tilde{K} \equiv \Pi^2/2m_d$ with $\tilde{x} \equiv (x - x_0)/l_H$ shifted by an imaginary amount $-ig$, so we can write it as :

$$\hat{K} = \exp(g l_H \hat{p}_x) \tilde{K} \exp(-g l_H \hat{p}_x). \quad (49)$$

The perturbation theory to Eq.(48) is most conveniently performed on the shifted ψ field defined by:

$$\psi(\tilde{x}) \equiv \exp(g l_H \hat{p}_x) \tilde{\psi}(\tilde{x}) = \tilde{\psi}(\tilde{x} - ig). \quad (50)$$

The transformed Hamiltonian is:

$$\tilde{H} \equiv \exp(-g l_H \hat{p}_x) (\hat{K} - \eta \hat{V}) \exp(g l_H \hat{p}_x)$$

Going to the creation and annihilation operators \hat{a}^\dagger and \hat{a} representation, Eq.(48) becomes

$$\left[\hat{a}^\dagger \hat{a} + \frac{1}{2} - \eta' \tilde{V}(\hat{a}, \hat{a}^\dagger) \right] \tilde{\psi}(x) = \frac{m_d}{e^* \hbar} \left(\omega_R + \alpha_d - \frac{1}{2} \gamma^2 m_d v^2 \right) \tilde{\psi}(x) \equiv \xi \tilde{\psi}(x). \quad (51)$$

Here

$$\tilde{V}(\hat{a}, \hat{a}^\dagger) = \exp \left[-i \frac{g}{\sqrt{2}} (\hat{a}^\dagger - \hat{a}) \right] (\hat{a}^{\dagger 2} + \hat{a}^2)^2 \exp \left[i \frac{g}{\sqrt{2}} (\hat{a}^\dagger - \hat{a}) \right]. \quad (52)$$

The "potential energy" can be further simplified using the identities:

$$\exp \left[-i \frac{g}{\sqrt{2}} (\hat{a}^\dagger - \hat{a}) \right] \hat{a} \exp \left[i \frac{g}{\sqrt{2}} (\hat{a}^\dagger - \hat{a}) \right] = \hat{a} + i \frac{g}{\sqrt{2}}, \quad (53)$$

$$\exp \left[-i \frac{g}{\sqrt{2}} (\hat{a}^\dagger - \hat{a}) \right] \hat{a}^\dagger \exp \left[i \frac{g}{\sqrt{2}} (\hat{a}^\dagger - \hat{a}) \right] = \hat{a}^\dagger + i \frac{g}{\sqrt{2}}. \quad (54)$$

It is helpful to note that the state resulting from the action of the shifting operator on $|0\rangle$ is a coherent state

$$|-i \frac{g}{\sqrt{2}}\rangle \equiv \exp \left[-i \frac{g}{\sqrt{2}} (\hat{a}^\dagger - \hat{a}) \right] |0\rangle. \quad (55)$$

The correction to the eigenvalue ξ (used later to find the phase transition boundary) to the first order η is then easily found:

$$\xi = \frac{1}{2} - \eta' (g^4 - 2g^2 + 2) \quad (56)$$

To the first order in η , the perturbed ground state is given by:

$$\tilde{\psi} = |0\rangle + \sum_{n=1}^4 \frac{\eta'}{n} |n\rangle \langle n| \left[\left(\hat{a}^\dagger + i \frac{g}{\sqrt{2}} \right)^2 + \left(\hat{a} + i \frac{g}{\sqrt{2}} \right)^2 \right]^2 |0\rangle \equiv |0\rangle + \eta' \sum_{n=1}^4 c_n |n\rangle. \quad (57)$$

where

$$\begin{cases} c_1 = -2\sqrt{2}ig(g^2 - 1) \\ c_2 = -2\sqrt{2}g^2 \\ c_3 = \frac{4\sqrt{3}}{3}ig \\ c_4 = \frac{\sqrt{6}}{2} \end{cases} \quad (58)$$

The solution of the original linearized TDGL equation Eq.(45) is simply the shifted $\psi(x)$ together with other factors:

$$d(x, y, t) = \exp[ik(y + vt)] \times \exp \left[-\frac{1}{2l_H^2} (x - kl_H^2 - igl_H)^2 \right] \times \left(\frac{1}{\pi l_H^2} \right)^{1/4} \left[1 + \eta \sum_{n=1}^4 c_n \frac{1}{\sqrt{2^n n!}} H_n \left(\frac{x}{l_H} - kl_H - ig \right) \right] \quad (59)$$

The solution we constructed is restricted to the case when the direction of the electric field is along the crystalline x direction. Now we generalize the calculation to arbitrary direction of the electric field. In the coordinate system fixed by the vortex lattice (which we will use for construction of the vortex lattice solution) the general vortex velocity is: $v_x = v \sin \theta, v_y = -v \cos \theta$, while the electric field is $E_x = E \cos \theta, E_y = E \sin \theta$. The angle between the crystal $[1,0,0]$ axis and the electric field will be therefore $\Theta = \theta - \varphi$ (see Fig. 4). The calculation is just a little bit more complicated. It still will be convenient to choose a coordinate system in which the direction of the electric field and that of the x axis coincide. The perturbed Hamiltonian then becomes:

$$\tilde{V}(\hat{a}, \hat{a}^\dagger) = \exp \left[-i \frac{g}{\sqrt{2}} (\hat{a}^\dagger - \hat{a}) \right] \left(e^{-2i(\theta-\varphi)} \hat{a}^{\dagger 2} + e^{2i(\theta-\varphi)} \hat{a}^2 \right)^2 \exp \left[i \frac{g}{\sqrt{2}} (\hat{a}^\dagger - \hat{a}) \right]$$

The corrected solution has the same form as Eq.(59), with the coefficients c_n which now depend on the angle $(\theta - \varphi)$, as follows:

$$\begin{cases} c_1 = -\sqrt{2}ig \left[\left(1 + e^{-4i(\theta-\varphi)}\right) g^2 - 2 \right] \\ c_2 = -\frac{\sqrt{2}}{2} \left(1 + 3e^{-4i(\theta-\varphi)}\right) g^2 \\ c_3 = \frac{4\sqrt{3}}{3}ig e^{-4i(\theta-\varphi)} \\ c_4 = \frac{\sqrt{6}}{2}e^{-4i(\theta-\varphi)} \end{cases} \quad (60)$$

The corresponding eigenvalue becomes

$$\xi = \frac{1}{2} - \eta' \left[\frac{1 + \cos 4(\theta - \varphi)}{2} g^4 - 2g^2 + 2 \right]. \quad (61)$$

B. Dependence of H_{c2} on the electric field

In the simpler case of electric field parallel to one of the crystal axes, from Eq.(56) the new phase boundary equation follows:

$$H_{c2} = H_{c2}^0 + \eta H_{c2}^1 = \frac{m_d}{e^*} (2\alpha_d - \gamma^2 m_d v^2) + \frac{2\eta m_d^3}{e^*} (5m_d^2 \gamma^4 v^4 - 12m_d \gamma^2 v^2 \alpha_d + 8\alpha_d^2)$$

where the second term is a perturbation. All the temperature dependence (at least within the confines of the simple Ginzburg - Landau assumption) is contained inside $\alpha_d = \alpha'(T_c - T)$. The phase transition line is therefore still quadratic in T ,

$$H_{c2}(T) = h_0 + h_1(T_c - T) + h_2(T_c - T)^2 \quad (62)$$

but first two coefficients have a nontrivial dependance on velocity v :

$$\begin{aligned} h_0 &= \frac{m_d^2}{e^*} \left(-1 + 10\eta m_d^3 \gamma^2 v^2 \right) \gamma^2 v^2 \\ h_1 &= 2\alpha' \frac{m_d}{e^*} \left(1 - 12\eta m_d^3 \gamma^2 v^2 \right) \\ h_2 &= 16\alpha'^2 \eta \frac{m_d^3}{e^*} \end{aligned} \quad (63)$$

Note that the curvature hasn't changed compared to the static case, but we have two new effects. First of all, the electric field (or, equivalently, electric current) shifts H_{c2} by a negative constant (proportional to E^2) to a lower value. This is expected. Secondly, although the

curvature h_2 doesn't change compared with the static case, the slope h_1 acquires a negative contribution proportional to E^2 .

In the general case of arbitrary orientation of the electric field, with respect to the crystal lattice, only the coefficient h_0 needs to be changed to:

$$h_0 = \frac{m_d^2}{e^*} \left\{ -1 + [9 + \cos 4(\theta - \varphi)] \eta m_d^3 \gamma^2 v^2 \right\} \gamma^2 v^2 \quad (64)$$

We got the interesting result that the shift in H_{c2} due to the electric field actually depends on the direction of the electric field relative to the crystal lattice. This result should be checked experimentally.

In the s - wave case (or $\eta = 0$) the boundary was first found and discussed in ref. [20]. There are a couple of peculiarities associated with it like the existence of a metastable normal state and the unstable superconductive state. The same applies to the present case. As far as we know, these peculiarities haven't been directly observed in low T_c materials. It would be interesting to reconsider this question for the high T_c materials. Note also that the phase transition is not a usual one (second order which probably turns to weakly first order due to fluctuations). In the presence of flux flow the two phases are stationary states rather than states in thermal equilibrium. There exist therefore a phase diagram in the space containing the current as an external parameter (both magnitude and direction).

C. Construction of the moving vortex lattice

Now we would like to follow a procedure similar to that described in Section IV.C for the static case to construct a vortex lattice solution. It turns out not to be a straightforward generalization. In earlier sections, we used the gauge freedom to make both the scalar and the vector potentials independent of y and t . This allows for separation of variables. The fact that y variable factored into the form $\exp(iky)$ helped us implement the periodicity in the y direction (with discrete values of k). However, in general, the vortex lattice will not be periodic along this special direction. To construct this general periodic solution, one has

to solve a very complicated periodicity constraint equation for the coefficients C_k , where k is now a continuous index.

In the static vortex lattice case, we used the gauge freedom to align the vector potential to the vortex lattice. This choice allows us to solve the constraint equation on C_k easily since we already had periodicity along the y -axis built in. This reduced the problem to a discrete one. Furthermore, only a few k_n 's were coupled, and it turned out to be solvable, at least for $p = 1$. This is not the case for the moving vortex lattice. In Subsection A, for the problem with electric field and time dependence, we used the gauge freedom to align the vector potential with respect to the electric field in order to find the general solution of the perturbed Hamiltonian. Now, when we have to use this general solution to construct the periodic solution we encounter the problem that we cannot use the gauge freedom to simultaneously simplify *both* problems. Fortunately, in the unperturbed (s-wave) case a simple Ansatz for the construction of moving vortex lattice solution exists. This works for the linearized TDGL equation with arbitrary direction of the electric field [20]. We shall use this observation to guide us in obtaining the periodic solution for the moving vortex lattice in the presence of perturbation. The solution can be explicitly checked to satisfy TDGL equation and the periodicity constraints.

In subsection A, we were forced to choose an axial gauge as in Eq.(46) (which will be referred to later as the gauge I),

$$\begin{aligned}\mathbf{A}^I(x', y', t) &= Hx'\hat{y}' \\ \Phi^I(x', y', t) &= \mathbf{v} \cdot \mathbf{A}^I = -vHx',\end{aligned}\tag{65}$$

For later convenience, we use x', y' to represent the coordinate in which the electric field is along x' direction, while x, y is the coordinate in which the vortex lattice is alligned to the y direction (see Fig.4). The relation is:

$$\begin{cases} x' = x \cos \theta + y \sin \theta \\ y' = -x \sin \theta + y \cos \theta \end{cases}$$

Fortunately we can transform our solution to a gauge in which the periodicity is manifest and the standard procedure works (referred as gauge II):

$$\begin{aligned}\mathbf{A}^{II}(x, y, t) &= H(x - v_x t) \hat{y} + \gamma \frac{m_d}{e^*} \mathbf{v} \times \hat{z} \\ \Phi^{II}(x, y, t) &= \mathbf{v} \cdot \mathbf{A}^B = v_y H(x - v_x t)\end{aligned}\quad (66)$$

where $v_x = v \sin \theta$, $v_y = -v \cos \theta$. The gauge transformation between two is determined by a phase $\chi(x, y, t)$ satisfying

$$\begin{aligned}\nabla \chi &= \mathbf{A}^I - \mathbf{A}^{II} \\ -\partial_t \chi &= \Phi^I - \Phi^{II}\end{aligned}$$

One of the solutions is:

$$\chi = \frac{\gamma m_d v}{e^*} x' + \frac{H}{2} \sin \theta \cos \theta \left[(y' + vt)^2 - x'^2 \right] + H \sin^2 \theta [x' (y' + vt)] \quad (67)$$

In this gauge unperturbed lattice can be easily formed using "boosted" solutions,

$$\Psi_n^{II}(x, y, t) = \frac{1}{\sqrt{L}} \left(\frac{1}{\pi l_H^2} \right)^{\frac{1}{4}} \exp[ik_n(y - v_y t)] \exp \left[-\frac{1}{2l_H^2} (x - v_x t - k_n l_H^2)^2 \right] \quad (68)$$

with standard coefficients: $\Psi^{II} = \sum_n C_n \Psi_n^{II}$. These elementary solutions are linearly related to the unperturbed normalized eigenfunctions in the gauge I found in Subsection V.A,

$$\Psi_k^I(x', y', t) = \frac{1}{\sqrt{L}} \left(\frac{1}{\pi l_H^2} \right)^{\frac{1}{4}} \exp \left(-\frac{g^2}{2} \right) \exp[ik(y' + vt)] \exp \left[-\frac{1}{2l_H^2} (x' - igl_H - kl_H^2)^2 \right] \quad (69)$$

after the gauge transformation is performed:

$$e^{ie^* \chi} \Psi_n^{II} = \frac{L}{2\pi} \int_{-\infty}^{\infty} dk B_{nk} \Psi_k^I$$

Note that gauge transformation and hence quantities B_{nk} are in general time dependent. However, we do not need to keep track of the time dependence here. The reason is that the GL equation we are solving with or without the perturbation is time translation invariant.

While keeping track of the time dependence of gauge non-invariant quantities B_{nk} and the wave functions is complicated, because in this case we would have to use a time dependent gauge choice, the gauge invariant quantities such as β_A are automatically time independent. Therefore, to simplify the calculation, we can set $t = 0$.

To find the coefficients B_{nk} two gaussian integrations should be performed:

$$B_{nk} = \int dx' dy' \left[\Psi_k^{I*}(x', y', 0) e^{ie^* \chi(x', y', 0)} \Psi_n^{II}(x, y, 0) \right] = \quad (70)$$

$$\frac{\sqrt{\pi} l_H}{L} \frac{1}{\sqrt{ie^{i\theta} \sin \theta}} \exp \left[-\frac{1}{2} \frac{\cos \theta}{\sin \theta} (k^2 + k_n^2) l_H^2 + ik l_H^2 \left(\frac{k_n}{\sin \theta} + \gamma m_d v \right) \right] \quad (71)$$

We have already found the first order correction to the wave function is gauge I. The corresponding expression in the gauge II is:

$$e^{ie^* \chi} \delta \Psi_n^{II} = \frac{L}{2\pi} \int_{-\infty}^{\infty} dk B_{nk} \delta \Psi_k^I$$

where

$$\begin{aligned} \delta \Psi_k^I &= \exp[ik(y' + vt)] \times \exp \left[-\frac{1}{2l_H^2} (x' - kl_H^2 - igl_H^2)^2 \right] \times \\ &\eta' \sum_{n=1}^4 c_n \frac{1}{\sqrt{2^n n!}} \left(\frac{H}{\pi} \right)^{1/4} H_n \left(\frac{x'}{l_H} - kl_H - ig \right) \end{aligned} \quad (72)$$

as was shown in the previous section.

It turns out that after a lengthy calculation, the correction to the wave function in gauge II is amazingly simple:

$$\begin{aligned} \delta \Psi_n^{II}(x, y) &= e^{-ie^* \chi} \frac{L}{2\pi} \int_{-\infty}^{\infty} dk B_{nk} \delta \Psi_k^I \\ &= \frac{1}{\sqrt{L}} \left(\frac{1}{\pi l_H^2} \right)^{1/4} \exp(ik_n y) \exp \left[-\frac{1}{2l_H^2} (x - k_n l_H^2)^2 \right] \\ &\times \eta' \sum_{m=1}^4 c_m \frac{e^{im\theta}}{\sqrt{2^m m!}} H_m \left(\frac{x}{l_H} - k_n l_H - ig e^{-i\theta} \right). \end{aligned} \quad (73)$$

An important observation is that the corrected moving lattice solution at $t = 0$

$$\begin{aligned} \Psi^{II}(x, y) &= \sum_{n=-\infty}^{\infty} C_n \left[\Psi_n^{II} + \delta \Psi_n^{II} \right] \\ &= \sum_{n=-\infty}^{\infty} C_n \frac{1}{\sqrt{L}} \left(\frac{H}{\pi} \right)^{1/4} \exp(ik_n y) \exp \left[-\frac{1}{2l_H^2} (x - k_n l_H^2)^2 \right] \times \\ &\left[1 + \eta' \sum_{m=1}^4 c_m \frac{e^{im\theta}}{\sqrt{2^m m!}} H_m \left(\frac{x}{l_H} - k_n l_H - ig e^{-i\theta} \right) \right] \end{aligned} \quad (74)$$

where $k_n = \frac{2\pi n}{a}$, and C_n again given by Eq.(29) is still invariant under the vortex lattice symmetries. From now on we work exclusively in gauge II. Physically this is understood as follows. Our perturbation commutes with all the translations generators, in particular with the translations defining the lattice structure. Therefore if the unperturbed function has certain lattice translation symmetry, the perturbed one will have as well.

D. The structure and magnetization of the moving lattice

The standard Abrikosov's procedure to develop an approximation for small order parameter around H_{c2} can be applied also in the flux flow case (see [20]). This time however the minimization of the Abrikosov parameter β_A does not correspond to minimization of energy, but rather to smallest deviation from the exact solution of TDGL equation. The "derivation" closely follows the static one. Using the expression for the vortex lattice solution found in the previous subsection, the correction term in expansion of the Abrikosov parameter β_A in η' ,

$$\beta_A = \beta_A^0 + \eta' \beta_A^1$$

is:

$$\beta_A^1 = \frac{\sqrt{\sigma}}{4} \text{Re} \left\{ \left[\sum_{n'} \exp(-2\pi i \zeta^* n'^2) \right] \left[\sum_n \exp(2\pi i \zeta n^2) G(n) \right] + \left(n \rightarrow n + \frac{1}{2}, n' \rightarrow n' + \frac{1}{2} \right) \right\}. \quad (75)$$

Here the function $G(n)$ is defined by

$$G(n) = e^{4i\varphi} (64\pi^2 \sigma^2 n^4 - 48\pi \sigma n^2 + 3) - 8e^{2i\varphi} g^2 \cos 2\Theta (8\pi \sigma n^2 - 1) \quad (76)$$

where $\Theta \equiv \theta - \varphi$ is the angle between the electric field and the crystal lattice. One immediately observes a surprising fact - the dependance on the angle Θ and velocity v is only via the combination $g^2 \cos 2\Theta$ where $g \equiv \gamma m_d v l_H$. It factors out as:

$$\beta_A(\varphi, \rho, \sigma) \equiv \beta_A^0(\rho, \sigma) + \eta' \operatorname{Re} \left[e^{4i\varphi} \delta(\rho, \sigma) + g^2 \cos 2\Theta e^{2i\varphi} \delta'(\rho, \sigma) \right]. \quad (77)$$

For example, the resulting lattice for $\Theta = \pi/4$ and arbitrary g will be the same as without electric field at all! Also apparent complete breaking of the rotational symmetry by general direction of the electric field is not felt by β_A . Indeed, lattice for some arbitrary Θ and g^2 is the same as for $\Theta = 0$ and $g^{2'} = g^2 \cos 2\Theta$. The fourfold symmetry has been reduced however. These results are nontrivial and can be checked experimentally. This degeneracy in velocity and Θ in the determination of vortex lattice may be a reflection of some dynamical symmetries which we have so far failed to see yet.

We get the $e^{\pm 2i\varphi}$ harmonics in Eq.(77) in addition to the fourth harmonic that appeared in the static case. The minimization with respect to φ still can be done analytically, although the algebraic equation in this case is quartic. For fixed η' and $g^2 \cos 2\Theta$, the minimization with respect to ρ and σ was performed numerically and we again obtain only rectangular body centered lattices aligned either to the crystalline axes. The angle α turns out to be only weakly dependent on the combination $g^2 \cos 2\Theta$. For example, for positive $\eta' = 0.015$, we obtain $\alpha = \alpha(g = 0) + 1.0g^2 |\cos 2\Theta|$ (in degrees) where $\alpha(g = 0) = 69.3^\circ$. The Abrikosov β_A is simply related to the slope of the magnetization curve

$$4\pi \frac{dM}{dH} = \frac{1}{(2\kappa^2 - 1)\beta_A} \quad (78)$$

as well as to other thermodynamic quantities. All of them therefore exhibit very simple dependence on the velocity \mathbf{v} , or, equivalently, on the current \mathbf{J} .

The fact that the optimal lattice is rectangular body centered is a bit mysterious. Rotational symmetry is completely broken by both the electric field and by the underlying crystal lattice. It is not easy to attribute the advantage of this lattice structure to some simple physical origin. It might be that it is a consequence of using the Abrikosov approximation, and therefore beyond this approximation the lattices might not be rectangular. Note also that it was also surprising that β_A was independent of the orientation of the electric field even in the s - wave calculation [20]. As far as we know, the preferred orientation of the moving lattice has not been observed in either low T_c or high T_c type II superconductors.

VI. THE NON-LINEAR CONDUCTIVITY NEAR H_{C2}

In this section we consider the dissipation in vortex cores due to flux flow. As it is well known, the fourfold symmetry forces the conductivity tensor σ_{ij} , defined by $J_i = \sigma_{ij}E_j$, to be rotationally symmetric. Namely, $\sigma_{ij} = \sigma\delta_{ij} + \sigma^H\varepsilon_{ij}$. Here σ is the usual (Ohmic) conductivity, σ^H is the Hall conductivity and ε_{ij} is the antisymmetric tensor. The additional term in the free energy corrects the values of σ and σ^H , but the correction is of the order η and therefore small. So, to see anisotropy, we definitely would like to go beyond linear response. This has been done for simple s - wave TDGL [20] near H_{c2} . We will neglect pinning and consider motion of a very large bundle. While there is a normal component of the conductivity, here we will concentrate on the contribution of the supercurrent only. For the discussion of the relative contribution of the two see [20].

A. Condensate for the moving lattice

To calculate the transport properties due to flux flow, we have to compute two quantities. The first is the expression for the electric current and will be obtained in the next subsection. The second is the normalization factor for the d - wave order parameter. In the previous sections we only needed Abrikosov's β_A which is insensitive to the overall scale of the condensate, but now we need to calculate it.

$$d = N \sum C_n(\Psi_n + \eta'\delta\Psi_n) \equiv N(\Psi + \eta'\delta\Psi) \cong (N_0 + \eta'N_1)(\Psi + \eta'\delta\Psi)$$

Here C_n , Ψ_n and $\delta\Psi_n$ have been calculated in the previous section, N is the normalization and N_0 is the normalization of Ψ . The calculation is standard. We again expand it to first order in η' . The normalization is determined from the minimization of the free energy as

$$\langle d^*d \rangle = \frac{\alpha_d}{2\beta} \frac{1}{\beta_A}$$

where $\langle \dots \rangle$ denote the space average, α_d and β are coefficients of the GL equation. The Abrikosov parameter β_A has its own η' expansion: $\beta_A = \beta_A^0 + \eta'\beta_A^1$ calculated in Section VI.C. Combining the two one obtains

$$N^2 \cong N_0^2(1 + \eta' \frac{2N_1}{N_0}) = \frac{\alpha_d}{2\beta} \frac{1}{\beta_A^0 \langle \Psi^* \Psi \rangle} \left[1 - \eta' \left(\frac{\beta_A^1}{\beta_A^0} + \frac{2 \operatorname{Re} \langle \Psi^* \delta \Psi \rangle}{\langle \Psi^* \Psi \rangle} \right) \right]$$

which will be used to calculate the current.

B. The direct and the Hall currents

The anisotropy term in the free energy Eq.(2) contains four covariant derivatives and consequently the electric current, in addition to the usual expression, contains additional terms. The leading order current is given by

$$\mathbf{J}^a = \frac{e^*}{2m_d} \langle d^* (\mathbf{\Pi} d) + (\mathbf{\Pi} d)^* d \rangle$$

The anisotropic perturbation to the current is

$$\begin{aligned} \mathbf{J}^b(d) = & e^* \eta \hat{\mathbf{x}}'' \left\langle \left[(\Pi_y'' - \Pi_x'') d \right]^* (\Pi_x'' d) + \left[\Pi_x'' (\Pi_y'' - \Pi_x'') d \right]^* d + c.c. \right\rangle \\ & - e^* \eta \hat{\mathbf{y}}'' \left\langle \left[(\Pi_y'' - \Pi_x'') d \right]^* (\Pi_y'' d) + \left[\Pi_y'' (\Pi_y'' - \Pi_x'') d \right]^* d + c.c. \right\rangle \end{aligned} \quad (79)$$

where $\Pi_x'' = \cos \varphi \Pi_x + \sin \varphi \Pi_y$, $\Pi_y'' = \cos \varphi \Pi_y - \sin \varphi \Pi_x$, and $\hat{\mathbf{x}}'' = \cos \varphi \hat{\mathbf{x}} + \sin \varphi \hat{\mathbf{y}}$, $\hat{\mathbf{y}}'' = \cos \varphi \hat{\mathbf{y}} - \sin \varphi \hat{\mathbf{x}}$ (See Fig. 4).

Substituting the condensate $d = N (\Psi + \eta' \delta \Psi)$ with Ψ and $\delta \Psi$ determined in the previous subsection, one obtains the expansion of \mathbf{J}^a to the first order in η' :

$$\begin{aligned} \mathbf{J}^a \cong & N^2 \left(\frac{e^*}{2m_d} \right) \{ [\langle \Psi^* \mathbf{\Pi} \Psi \rangle + \langle \Psi \mathbf{\Pi} \Psi^* \rangle] + \\ & \eta' [\langle \Psi^* \mathbf{\Pi} \delta \Psi \rangle + \langle \delta \Psi^* \mathbf{\Pi} \Psi \rangle + \langle \delta \Psi \mathbf{\Pi} \Psi^* \rangle + \langle \Psi \mathbf{\Pi} \delta \Psi^* \rangle] \} \\ \equiv & \mathbf{j} + \eta' (\delta \mathbf{j}_1 + \delta \mathbf{j}_2) \end{aligned}$$

where $\delta \mathbf{j}_1$ comes from the correction to N^2 , and $\delta \mathbf{j}_2$ contains the correction to the wave function $\delta \Psi$.

$$\mathbf{j} = N_0^2 \left(\frac{e^*}{2m_d} \right) [\langle \Psi^* \mathbf{\Pi} \Psi \rangle + \langle \Psi \mathbf{\Pi} \Psi^* \rangle] = \left(\frac{e^* \alpha_d}{4\beta m_d} \frac{1}{\beta_A^0} \right) \frac{2 \operatorname{Re} \langle \Psi^* \mathbf{\Pi} \Psi \rangle}{\langle \Psi^* \Psi \rangle} \quad (80)$$

$$\delta \mathbf{j}_1 = - \left(\frac{\beta_A^1}{\beta_A^0} + \frac{2 \operatorname{Re} \langle \Psi^* \delta \Psi \rangle}{\langle \Psi^* \Psi \rangle} \right) \mathbf{j}$$

$$\delta \mathbf{j}_2 = \left(\frac{e^* \alpha_d}{4\beta m_d} \frac{1}{\beta_A^0} \right) \frac{4 \operatorname{Re} \langle \delta \Psi^* \Pi \Psi \rangle}{\langle \Psi^* \Psi \rangle} \quad (81)$$

The expansion to first order in η' of \mathbf{J}^b is:

$$\mathbf{J}^b(N\Psi) \simeq N_0^2 \mathbf{J}^b(\Psi) \equiv \eta' \delta \mathbf{j}_3$$

$$\begin{aligned} \delta \mathbf{j}_3 = & \left(\frac{e^* \alpha_d}{2\beta} \frac{1}{\beta_A^0} \right) \left(\frac{l_H^2}{m_d} \right) \times \\ & \left\{ \hat{\mathbf{x}}'' \frac{\langle [(\Pi_y''^2 - \Pi_x''^2) \Psi]^* (\Pi_x'' \Psi) + [\Pi_x'' (\Pi_y''^2 - \Pi_x''^2) \Psi]^* \Psi + c.c. \rangle}{\langle \Psi^* \Psi \rangle} \right. \\ & \left. - \hat{\mathbf{y}}'' \frac{\langle [(\Pi_y''^2 - \Pi_x''^2) d]^* (\Pi_y'' d) + [\Pi_y'' (\Pi_y''^2 - \Pi_x''^2) d]^* d + c.c. \rangle}{\langle \Psi^* \Psi \rangle} \right\} \end{aligned}$$

Summing up the corrections, one obtains the corrected current as :

$$J = J^a + J^b = \mathbf{j} + \eta' \delta \mathbf{j} = \mathbf{j} + \eta' (\delta \mathbf{j}_1 + \delta \mathbf{j}_2 + \delta \mathbf{j}_3)$$

Here we have used the fact that the operator Π is Hermitian, and the curly bracket denotes anticommutator. Note that although the total wave function is a linear combination of $\Psi_n(x, y)$, after averaging over the 2D space all the components decouple due to the $\exp(ik_n y)$ factors and the fact that the current is quadratic in Ψ_n . This means that it suffices to consider only one of the components to calculate \mathbf{J} . The results for the three contributions to the current correction are:

$$\frac{-2 \operatorname{Re} \langle \Psi^* \delta \Psi \rangle}{\langle \Psi^* \Psi \rangle} \mathbf{J} = \frac{e^* \alpha_d}{\beta} \frac{1}{\beta_A^0} \left\{ -g^4 \left[1 + \frac{2}{3} \cos 4\Theta \right] + 4g^2 \right\} (\gamma \mathbf{v} \times \hat{\mathbf{z}}) \quad (82)$$

$$\frac{e^* \alpha_d}{4\beta m_d} \frac{1}{\beta_A^0} \frac{4 \operatorname{Re} \langle \delta \Psi^* \Pi \Psi \rangle}{\langle \Psi^* \Psi \rangle} = \frac{e^* \alpha_d}{\beta} \frac{1}{\beta_A^0} \left\{ g^4 \left[1 + \frac{2}{3} \cos 4\Theta \right] - 4g^2 - 2 \right\} (\gamma \mathbf{v} \times \hat{\mathbf{z}}) \quad (83)$$

$$\begin{aligned} & \frac{e^* \alpha_d}{2\beta} \frac{l_H^2}{m_d} \frac{1}{\beta_A^0} \left\{ \hat{\mathbf{x}}'' \frac{2 \operatorname{Re} \langle \Psi^* \{ \Pi_y''^2 - \Pi_x''^2, \Pi_x'' \} \Psi \rangle}{\langle \Psi^* \Psi \rangle} - \hat{\mathbf{y}}'' \frac{2 \operatorname{Re} \langle \Psi^* \{ \Pi_y''^2 - \Pi_x''^2, \Pi_y'' \} \Psi \rangle}{\langle \Psi^* \Psi \rangle} \right\} \\ & = \frac{e^* \alpha_d}{\beta} \frac{1}{\beta_A^0} \left\{ \left[g^2 (1 + \cos 4\Theta) + 2 \right] (\gamma \mathbf{v} \times \hat{\mathbf{z}}) - g^2 \sin 4\Theta (\gamma \mathbf{v}) \right\} \end{aligned} \quad (84)$$

where $\Theta \equiv \theta - \varphi$ as before. Summing them up we got our final expression:

$$\delta \mathbf{j} = -\frac{\beta_A^1}{\beta_A^0} \mathbf{j} + \frac{e^* \alpha_d}{\beta} \frac{1}{\beta_A^0} g^2 (1 + \cos 4\Theta) (\gamma \mathbf{v} \times \hat{\mathbf{z}}) - \frac{e^* \alpha_d}{\beta} \frac{1}{\beta_A^0} g^2 \sin 4\Theta (\gamma \mathbf{v})$$

From this, one obtains the simple results Eq.(4,5) advertised earlier. Note that all the g^4 terms are cancelled.

VII. CONCLUDING REMARKS

Instead of summarizing the results (which has been done in Section I), we briefly comment on the possibility of observation of various phenomena quantitatively discussed in this paper.

.1. Internal structure of a single anisotropic vortex Although direct observation of the order parameter using scanning - tunneling-microscopy (STM) [32], or the magnetic field distribution [16] using electron holography [29] or other techniques is possible, the detailed effects hotly debated by theoreticians (where the zeroes of the s field are located, small distance asymptotics) probably do not have a significant impact on such experiments. One also should note that the Ginzburg - Landau framework adopted here might not be applicable close to the vortex center where microscopic excitation spectrum becomes important. An approach using elements of the microscopic theory (via Bogoliubov - DeGennes equations along the lines of [33] and [15]) will be necessary.

.2. Structure of static and moving anisotropic vortex lattice Static vortex lattice has been observed using small angle neutron scattering [6] and tunneling spectroscopy [7]. Although moving vortices have been directly observed using electron tomography [27], to our knowledge the shape and orientation of moving large bundles hasn't been observed as yet. Moving vortex lattice is much more sensitive to pinning effects than the static lattice. In the static case pinning can just slightly distort or cause breakup of the crystal to smaller peaces. For the moving flux lattice pinning is expected to be much more significant. The

orientation effect that we predict is very small, but the asymmetry in magnetization might be quite significant.

We found the transition point for the parameter $\eta' = \eta m_d e^* H$ is at $\eta'_c = 0.0235$. This transition between the rectangular and the square lattices might be seen in neutron scattering experiments, since the square lattice has higher symmetry (number of spots is reduced to four at the transition). Note that by increasing magnetic field the critical η'_c can be exceeded without changing the sample (η is independent of magnetic field).

.3. Static transition to the normal phase As is well known, in the presence of fluctuations, the second order phase transition from superconducting to normal state becomes a weakly first order melting line into the vortex liquid. This is the reason that the present study of the diagram will be useless for BSCCO which has a relatively large Ginzburg number. For YBCO and the low temperature superconductors, the curvature of the phase transition line can in principle provide an estimate of η with reservations mentioned in the end of Subsection IV.C.

.4. Dynamic phase diagram Dynamical phase diagram, namely transition from the moving lattice to normal (or moving liquid) should be complicated by pinning effects. However, provided these could be overcome a number of interesting effects could be observed. First, even neglecting the rotational symmetry breaking effects, there are a number of peculiarities associated with normal - superconducting boundary noted by Thompson and Hu like the existence of a metastable normal state and the unstable superconductive state. As far as we know, these peculiarities haven't been convincingly observed [34] in low T_c materials. It would be interesting to reconsider this question for the high T_c materials.

In the s - wave case however, the phase diagram cannot depend on orientation of the current. We calculated this orientation dependence on the angle between the atomic lattice and the direction of current or electric field to first order in η , see Eq.(62,63,64). New effects include the change in slope of H_{c2} as function of temperature, not only in curvature.

a. Nonlinear I-V curves and magnetization One should be able to measure currents in the same sample oriented differently with respect to the atomic crystal. Note that the effect can be seen in low temperature anisotropic superconductors, not necessarily in YBCO. The simplicity of the expressions for both direct and Hall current Eqs.(4,5) calls for some special ways to verify it experimentally. The angular dependence of the magnetization near the transition (given by Eqs.(78,75)) might be large enough to be measurable.

There are number of limitations of our approach which can be lifted by possible extensions. One of them is the assumption of exact fourfold symmetry. Deviations from it in a form of different coefficients of the gradient terms in x and y directions have already been studied recently [35] using the two field formalism. If they happen to be small, they can be easily added perturbatively. These effects of explicit breaking are clearly quite different from those of spontaneous breaking of the fourfold symmetry studied here. Our results for the lattices are limited to fields close to H_{c2} only. It is possible, although more difficult to extend them to lower magnetic fields. Another interesting direction is the influence of the anisotropy on vortex fluctuations in the lattice. We hope to address these issues in the future.

Also anisotropy influences fluctuations not considered here. In addition, the effective one component approach allows to consider possibilities not apparent within the two field one. For example, the coefficient η , in principle, can be negative despite the fact that within the two field formalism it should be positive. Twinning is expected to reduce the value of the parameter.

APPENDIX: COMPARISON OF THE TWO FIELD AND THE ONE FIELD RESULTS FOR SINGLE VORTEX

In the two field formulation, the small r asymptotics of the solution for the d - wave component of the isolated vortex is given by :

$$d(r, \phi) \simeq (d_1 r + d_3 r^3) e^{i\phi} \tag{A1}$$

where the subleading term coefficients is

$$d_3 = -\frac{d_1}{8\xi_d^2} \left[1 + \frac{h_0}{H_{c2}(0)} \right] \simeq -\frac{d_1}{8\xi_d^2},$$

neglecting terms proportional to $h_0/H_{c2}(0)$ [13]. The s -component asymptotics is:

$$\begin{aligned} s(r, \phi) &\simeq -\frac{\gamma_v}{\alpha_s} (\Pi_y^2 - \Pi_x^2) d(r) \\ &\simeq -\frac{\gamma_v}{\alpha_s} (4d_3 r e^{-i\phi}) = +\frac{1}{2} \left(\frac{\gamma_v}{\alpha_s \xi_d^2} \right) d_1 r e^{-i\phi} \end{aligned} \quad (\text{A2})$$

The expression Eq.(A2) is different from what was obtained in [13], Eq. (22), but qualitatively the behavior is not affected. We also found that it doesn't follow from their Eq. (19), because to the same order of approximation [13] had a non-vanishing term proportional to $e^{3i\phi}$. Nevertheless the concluding statement in [13] is basically correct. Following the same argument leading to an estimate of the maximal amplitude of $s(r)$ as in [13] we obtained

$$\frac{s_{\max}}{d_0} \simeq \frac{1}{4} \left(\frac{\gamma_v}{\alpha_s \xi_d^2} \right)$$

apparently this correction accounts for the 20% error cited in [13].

The asymptotic form of the wave function was used to make a topological argument about poles in the s - wave. Due to the different winding number of small r and large r asymptotics of $s(r, \phi)$, there must exist four poles in the intermediate region. This was shown numerically in [13]. Ting *et al* [17] however performed a similar calculation, but didn't get the poles. Our calculation, which is much simpler than the two field one, confirms the former and shows clearly four poles on the x and y axis, independent of what kind of approximate d -component wave function one chooses. We suspect that the numerical simulation in [17] was not sensitive enough to resolve these poles [36].

APPENDIX: ACKNOWLEDGMENTS

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Figure Captions

FIG. 1 A single vortex solution of the one component GL equation. The coefficient function f_1, f_{-3}, f_5 for the harmonics $e^{i\phi}, e^{-3i\phi}, e^{5i\phi}$, respectively. f_i 's are given in units of $\Psi_0 = \alpha_d/2\beta$, and r is given in units of ξ_d . See Eq. (13),(14),(15)

FIG. 2 The d -field of a single vortex for $\eta = 0.15$. Only the absolute value of the d field in units of Ψ_0 is shown. (a) Contour plot. (b) Three dimensional plot.

FIG. 3 The s -field of a single vortex for $\eta = 0.15$. (a) Contour plot. (b) Three dimensional plot. Note that there are four singularities on which the s -field vanishes.

FIG. 4 The coordinate system used in our calculations, this defines the angles θ, φ and Θ .

FIG. 5 The Abrikosov parameter β_A as a function of the lattice parameters (ρ, σ) (defined in Eq.(40)). There are three degenerate local minima. Oblique lattices are on the lines $\rho = 1/2$ and $\rho^2 + \sigma^2 = 1$. The two points A and B are related by $\rho \rightarrow 1/\rho$ and therefore represents the same lattice. Point C represents the same rectangular lattice rotated by 90° .

FIG. 6 The angle α as a function of η' , the two branches correspond to lattices related by a rotation of 90° . A continuous transition from the rectangular lattice to the square lattice happens at $\eta'_c = 0.235$.

FIG. 7 The Abrikosov parameter β_A as a function of η' for triangular, square and optimal rectangular body centered lattices, respectively. At the transition point η'_c , the rectangular lattice is taken over by the square lattice. Note that η' is proportional to the magnetic field H .

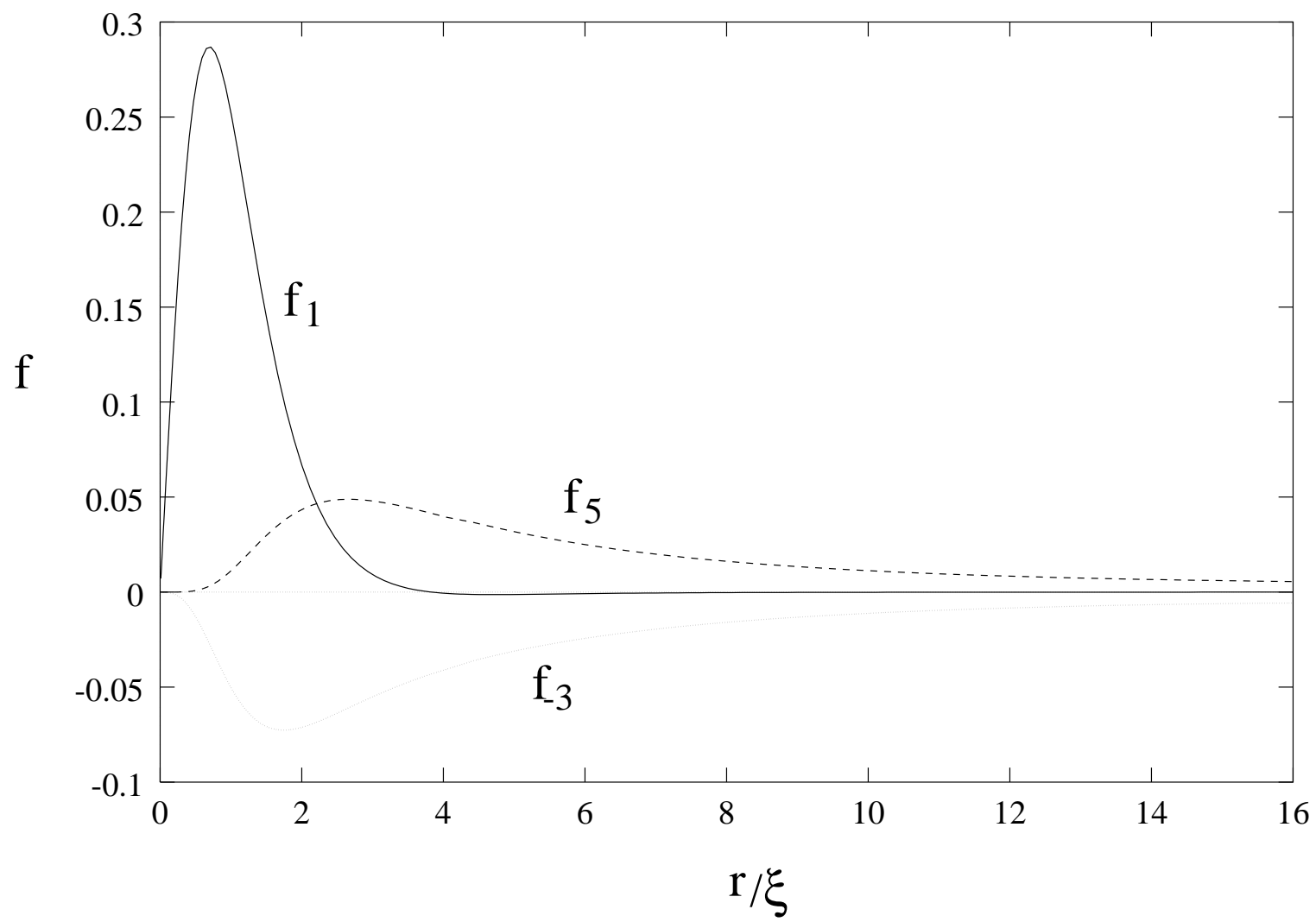


Fig. 1

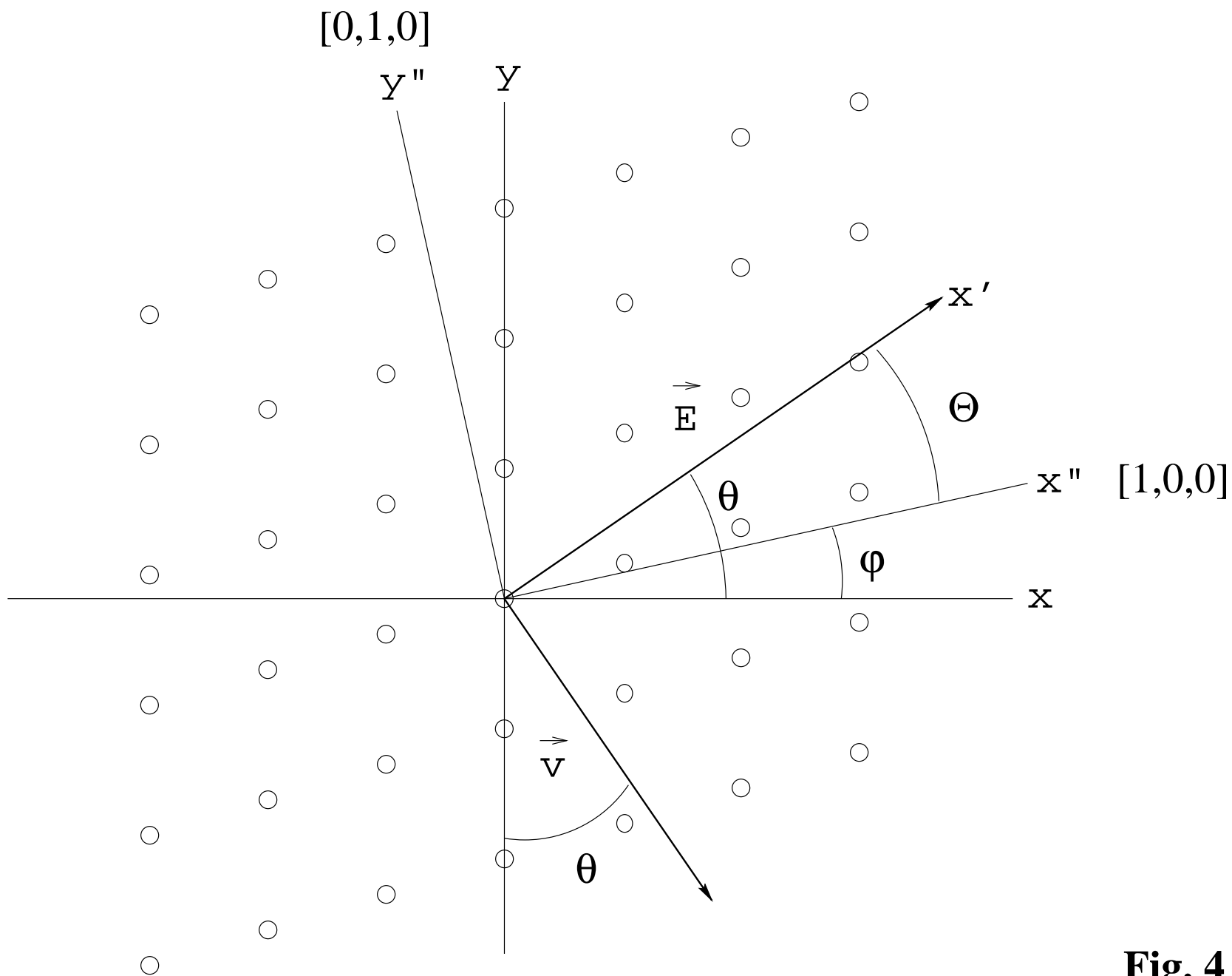


Fig. 4

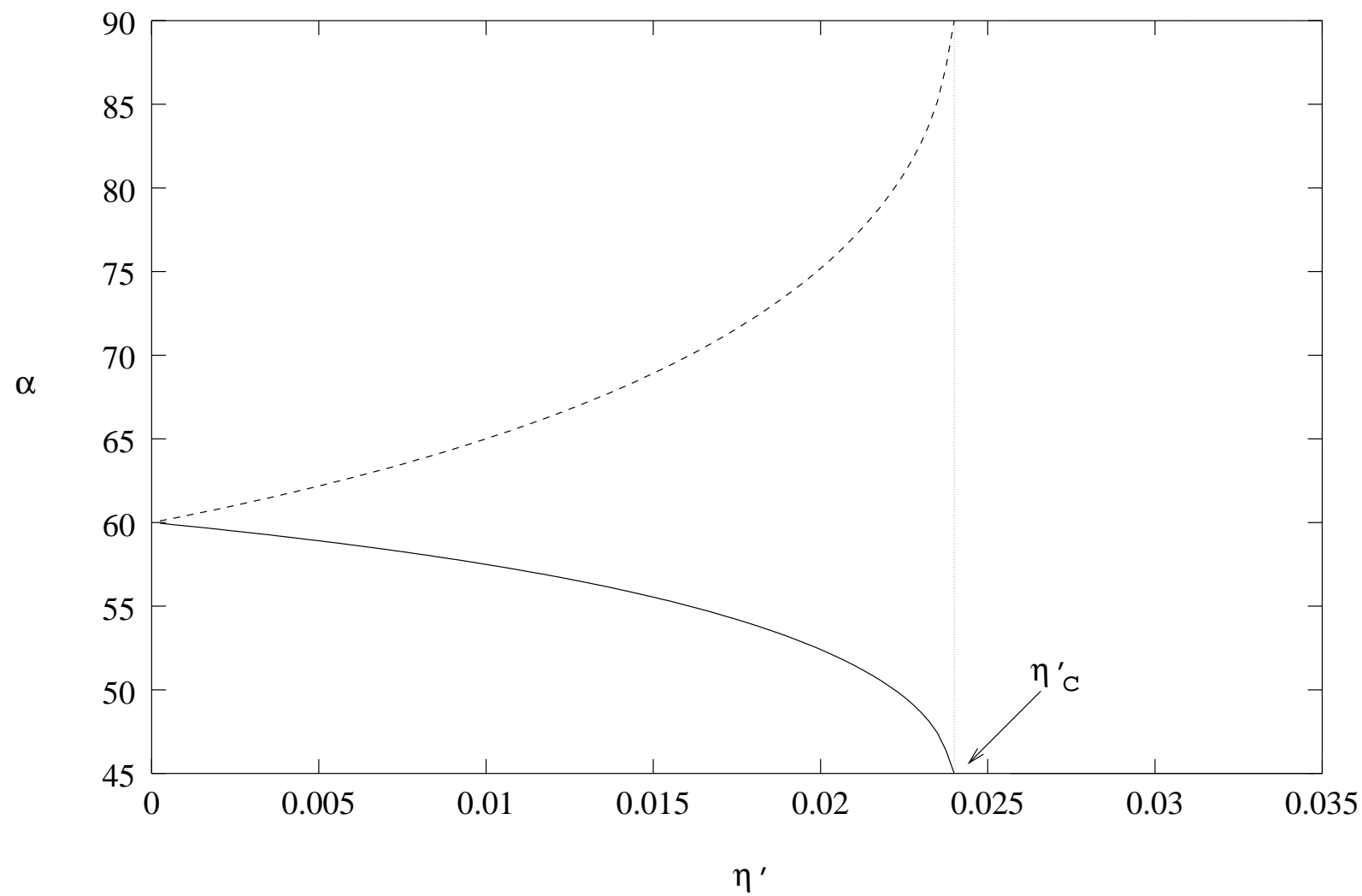


Fig.6

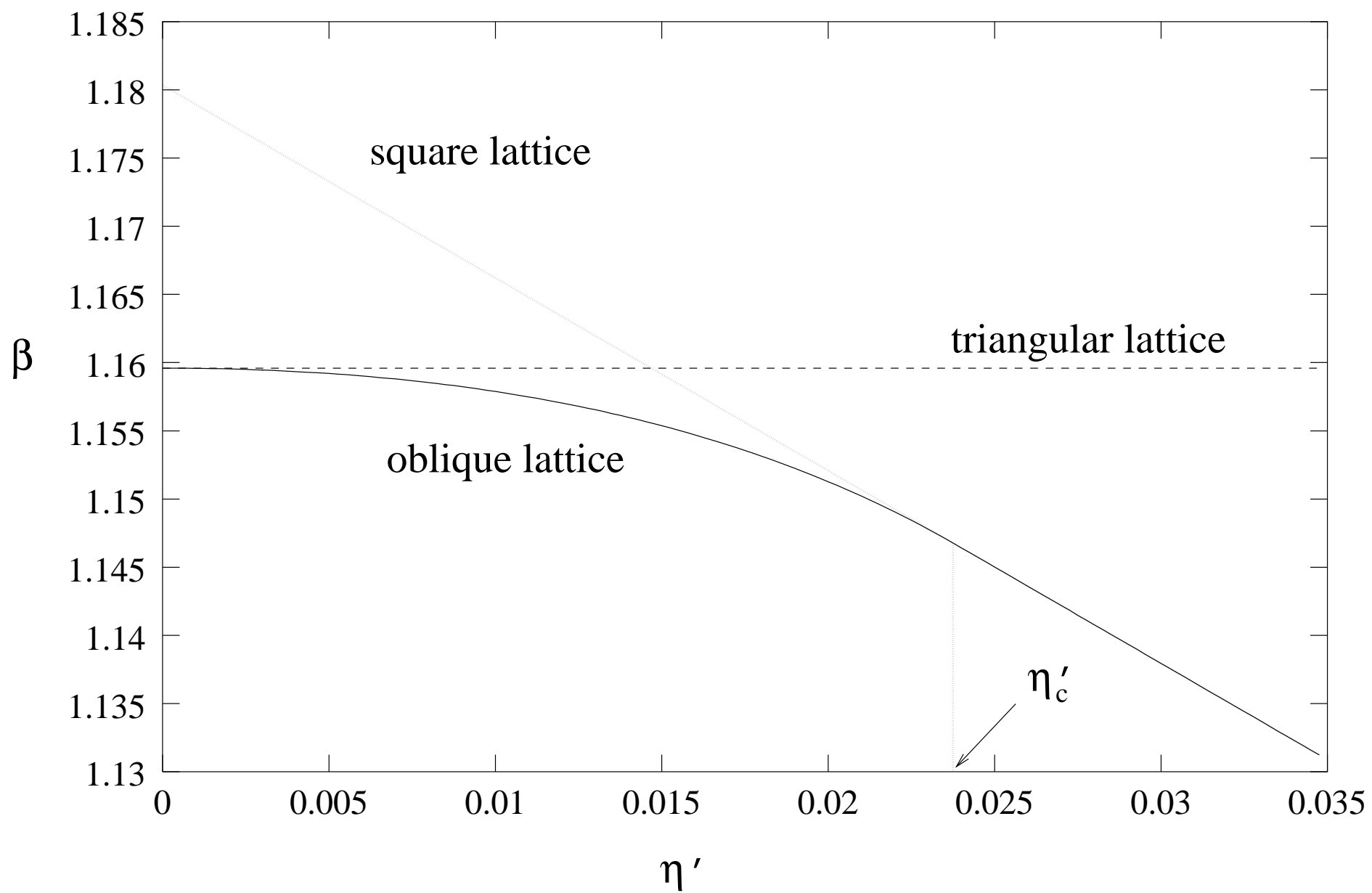


Fig. 7